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*Go forth, my little book! pursue thy way!*

*Go forth, and please the gentle and the good.*

William Wordsworth
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Here is my journey’s end.
William Shakespeare (Macbeth)